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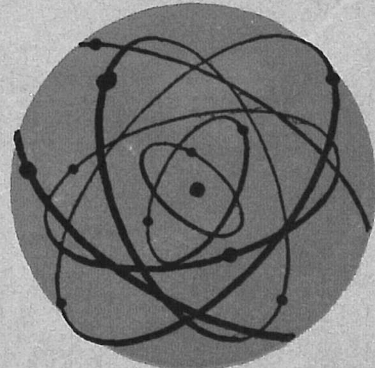
Increasing space-charge waves



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Increasing Space-Charge Waves

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An earlier paper presented equations for increasing waves in the presence of two streams of charged particles having different velocities, and solved the equations assuming the velocity of one group of particles to be zero or small. Numerical solutions giving the rate of increase and the phase velocity of the increasing wave for a wide range of parameters, covering cases of ion oscillation and double-stream amplification, are presented here.

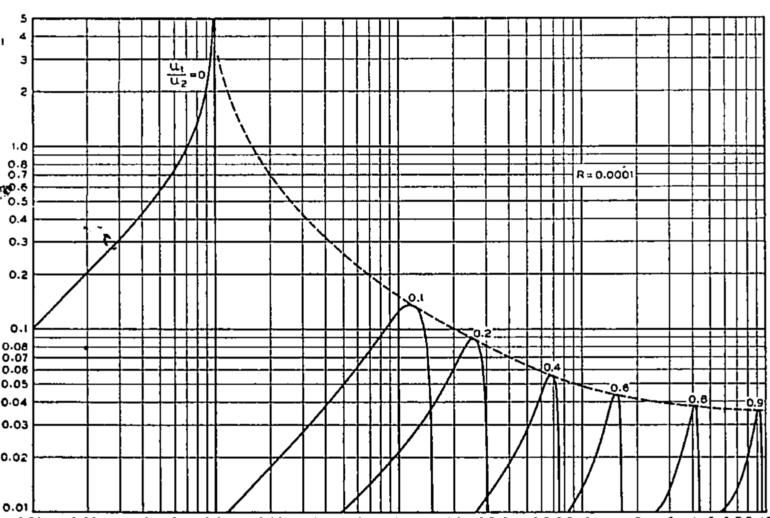
In an earlier paper[1] the writer discussed the increasing waves which can exist in an electron flow made up of charged particles of two or more velocities. This original discussion was concerned with such waves as a cause of fluctuations in long electron streams. Subsequently, however, there has been a considerable interest in waves of this type from other points of view, both as a source of radio-frequency components of solar radiation,[2,3] and as a means for amplifying radiofrequency signals.[3–7] The equations given earlier[1] involve as parameters the velocities of the two or more components of space-charge and their plasma frequencies, but not the signs of the charges. Thus, these equations are the same as, or are simplified forms of, those given later.[3–7] The only difference is that a different range of parameters is involved in various problems. In the case of ion oscillations in electron beams, the electrons have a much higher plasma frequency than the ions and are moving much faster than the ions. In obtaining amplification using two electron streams, the velocities and plasma frequencies of the two streams can be nearly equal. It is not clear to the writer what range of parameters would be involved in the case of solar radiation.

Because of the increasing interest in increasing space-charge waves, it seems of some value to explore the nature of such waves for a wide range of parameters, embracing at least the range of ion noise and of double-stream radio frequency amplification, in order to compare behavior in various ranges. This is the purpose of the present paper.

We will start with (18) of reference 1.

$$1=\frac{\omega_e^2}{(\omega+j\Gamma u_0)^2}+\frac{\omega_i^2}{\rho_0}\int_{u=-\infty}^{u=\infty}\frac{d\rho_0}{(\omega+j\Gamma u)^2}. \qquad (1)$$

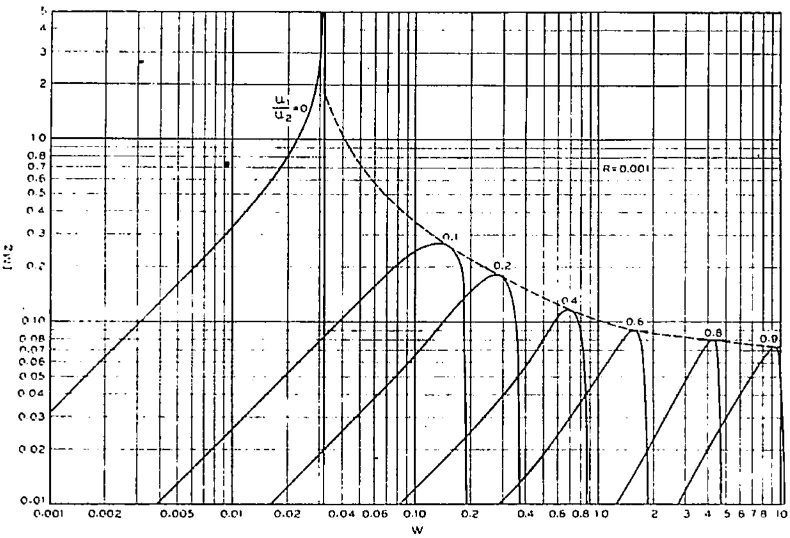
This equation applies to a one-dimensional or parallel-



Figs. 1–6 and 14. The gain parameter, Imz, *vs.* the ratio W of the frequency to the plasma frequency of the faster stream for a given ratio R of the squares of the plasma frequencies of the slower and faster streams and for various ratios, u_1/u_2, of velocity of the slower stream to velocity of the faster stream. The wave amplitude varies with distance x as $\exp[Imz(\omega_{02}/u_2)x]$, where ω_{02} is the radian plasma frequency of the faster stream.

[1] J. R. Pierce, "Possible fluctuations in electron streams due to ions," J. App. Phys. **19**, 231–236 (1948).
[2] V. A. Bailey, Nature **16**, 599 (1948), J. Roy. Soc. NSW **82**, 107 (1948), Inst. J. Sci. Res. **A1**, 351–359 (1948), proposes a somewhat different type of increasing wave, that in an ion cloud in the presence of an electric field, as an explanation of solar radiation components.
[3] A. V. Haeff, "Space-charge wave amplification effects," Phys. Rev. **74**, 1532–1533 (1948).
[4] J. R. Pierce and W. B. Hebenstreit, "A new type of high-frequency amplifier," Bell Sys. Tech. J. **28**, 33–51 (1949).
[5] A. V. Hollenberg, "Experimental observation of amplification by interaction between two electron streams," Bell Sys. Tech. J. **28**, 52–58 (1949).
[6] A. V. Haeff, "The electron-wave tube—a novel method of generation and amplification of microwave energy," Proc. IRE (7, 4–10 (1949).
[7] L. S. Nergaard, "Analysis of a simple model of a two-beam growing-wave tube," RCA Review **9**, 585–601 (1948).

FIG 2.



plane case, and applies to a small sinusoidal disturbance which is assumed to vary with time and distance in the x direction, the direction of flow, as $\exp(j\omega t - \Gamma x)$. Here ω_e is the electron plasma radian frequency and u_0 is the mean velocity of the electrons, while ω_i is the ion plasma radian frequency based on the total ionic charge density ρ_0, and $d\rho_0$ is an element of charge density associated with ions of velocity u.

We will consider a case of particles (electrons or ions) of two velocities, u_1 and u_2, only. Particles of these two velocities will have average charge densities $\pm\rho_{01}$ and $\pm\rho_{02}$, and charge-to-mass ratios $\pm(q/m)_1$, $\pm(q/m)_2$. Thus, the plasma radian frequencies ω_{01}^2 and ω_{02}^2 will be given by

$$\omega_{01}^2 = \frac{(q/m)_1\rho_{01}}{\epsilon_0}, \tag{2}$$

$$\omega_{02}^2 = \frac{(q/m)_2\rho_{02}}{\epsilon_0}. \tag{3}$$

Because Γ is nearly a pure imaginary, and in accord with the notation used in a later paper,[4] another variable β will be introduced

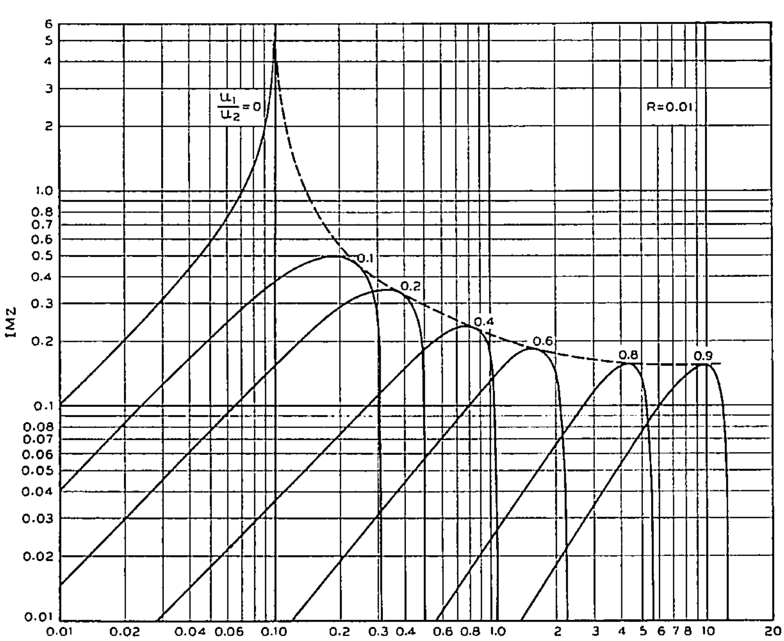
$$j\beta = \Gamma. \tag{4}$$

There are now particles of a velocity u_2 only involved in the integration in the last term of (1), and in the notation of (2)–(4), (1) becomes

$$1 = \frac{\omega_{01}^2}{(\omega - \beta u_1)^2} + \frac{\omega_{02}^2}{(\omega - \beta u_2)^2}. \tag{5}$$

This equation will now be rewritten in terms of three

FIG. 3.



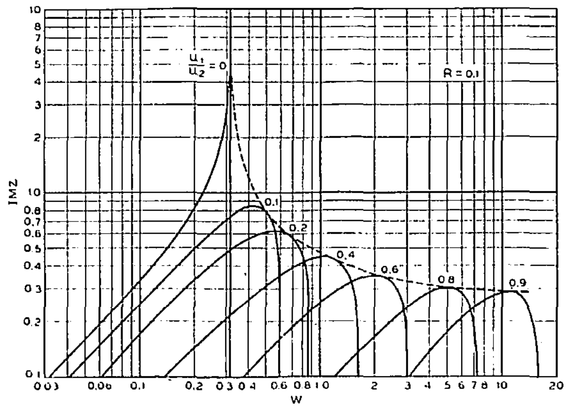


FIG. 4.

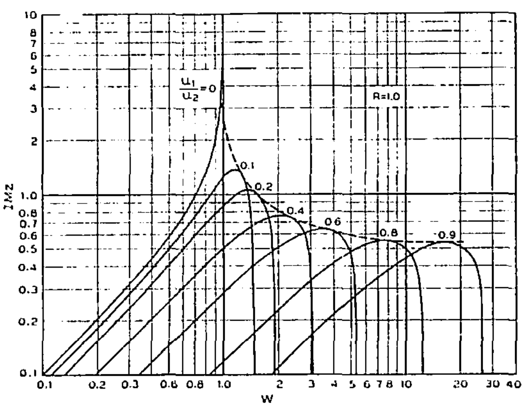


FIG. 5.

parameters chosen to have a simple physical significance in that they refer quantities to the plasma frequency and the velocity of the faster stream of particles.

$$R=(\omega_{01}/\omega_{02})^2. \tag{6}$$

Thus, R is merely the ratio of the squares of the plasma frequencies. For particles of the same charge-to-mass ratio, R is the ratio of the charge densities.

$$W=\omega/\omega_{02}. \tag{7}$$

Thus, W is merely the ratio of the frequency to the plasma frequency of one group of particles, which we shall choose to be the faster group.

The third parameter is the ratio of the mean velocities (u_1/u_2). As we have taken u_2 to be the velocity of the faster group of particles, $(u_1/u_2)<1$.

We will also choose a new variable z in place of β

$$z=\beta/(\omega_{02}/u_2). \tag{8}$$

The amplitude of the disturbance increases with distance as

$$\exp[Imz(\omega_{02}/u_2)x] \tag{9}$$

where Imz is the imaginary part of z (a real number). Thus, the rate of increase depends on the plasma frequency and the velocity of the faster stream, and on the parameter z.

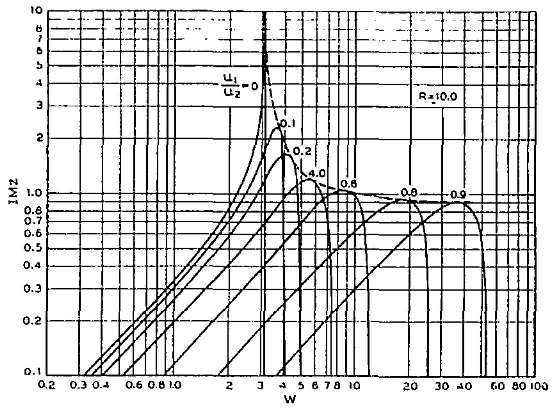


FIG. 6.

The real part of β is by definition the radian frequency divided by the phase velocity, v. Hence,

$$v=\left(\frac{W}{Rez}\right)u_2. \tag{10}$$

In terms of the new parameters and variable, (5) becomes

$$1=\frac{R}{(W-(u_1/u_2)z)^2}+\frac{1}{(W-z)^2}. \tag{11}$$

This equation has been solved numerically for a range of values of the parameters.

Let us consider first the behavior of gain with frequency. This is specified by the parameter Imz. The quantity $(\omega_{02}/u_2)x$ is a measure of distance defined in terms of the plasma frequency and the velocity of the faster stream and distance x. Thus, Imz is a measure of gain per unit distance, and if we take u_2/ω_{02} as a unit distance, the gain in this unit distance is $\exp(Imz)$.

In Figs. 1–6, Imz is plotted *vs.* W, the ratio of the frequency to the plasma frequency of the faster stream, for ratios of the squares of the plasma frequencies of the slower and faster streams, $R=0.0001$, 0.001, 0.1, 1, 10. The smallest of these values could represent the presence of gas ions in a charge density comparable to the charge density of a faster electron stream. Or, it could represent the presence of slower electrons forming a current of less than a ten-thousandth of the current carried by the fast electrons.

When the slower electrons are actually at rest, (11) assumes the form

$$1=\frac{R}{W^2}+\frac{1}{(W-z)^2}, \tag{12}$$

$$z=W\pm j(R/W^2-1)^{-\frac{1}{2}}$$

We see that at low frequencies the gain rises with frequency and goes to infinity at

$$W=(R)^{\frac{1}{2}}. \tag{13}$$

At higher frequencies z is real and there is no gain.

When the slow electrons are actually moving, the

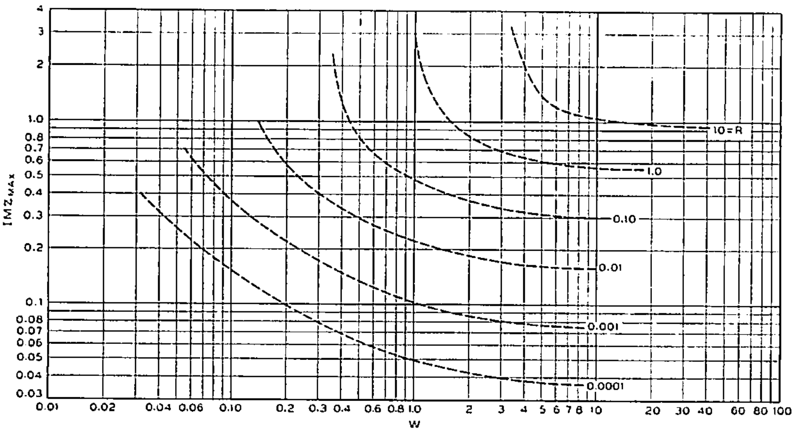
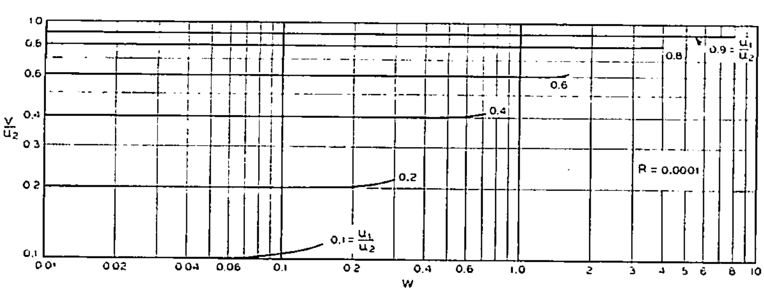


FIG. 7. The envelopes of the curves of Figs. 1–6, showing the largest values of Imz *vs.* W.

maximum gain is finite and occurs at frequencies higher than $(R)^{1/2}$, the frequency of maximum gain increasing as u_1/u_2 is made larger and the velocity difference becomes smaller. As the velocities approach equality the frequency of maximum gain approaches infinity and the maximum gain approaches a constant value which is a function of R.

For any velocity ratio u_1/u_2, at low frequencies W the gain per unit distance, (Imz), is proportional to frequency. At low frequencies the actual current is many times the minimum current necessary to produce double-stream gain, and the gain per wave-length should approach an asymptotic value, at least near $R=1$,[4] and hence the gain per unit distance should be proportional to frequency. Examining Fig. 5 ($R=1$) we see that this is so. Moreover, the maximum gain per wave-length (attained at low frequencies) should be proportional to the fractional velocity separation. We see from Fig. 5 that at low frequencies (say, $W=2$) going from $u_1/u_2=0.9$ to $u_1/u_2=0.8$, which about doubles the fractional velocity separation, doubles the gain also.

For a given value of R and u_1/u_2, the gain per unit



FIGS. 8–13 and 15. The ratio of wave phase velocity v to the velocity of the faster stream, u_2 *vs.* the ratio W of the frequency to the plasma frequency of the faster stream.

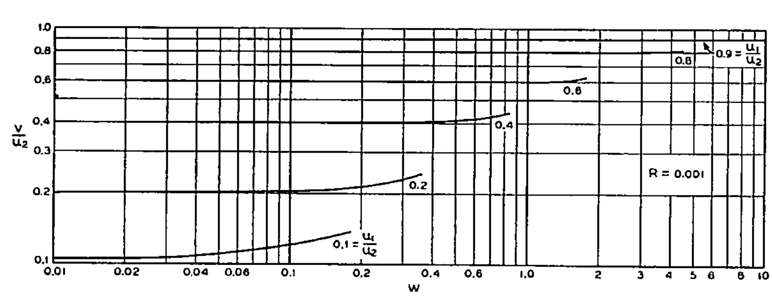


FIG. 9.

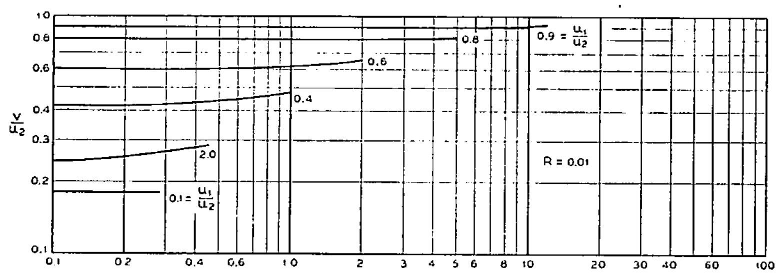


FIG. 10.

distance first rises as frequency is increased, then reaches a maximum value, and finally falls to zero at some frequency at which the actual current is just equal to the current necessary to produce an increasing wave. At higher frequencies all four values of z are real and represent unattenuated waves.

One of the surprising features of the curves of Figs. 1–6 is the small differences caused by large changes in R. For instance $R_1=0.0001$ might represent a ten-thousandth as many low velocity electrons as high-velocity electrons, while $R=10$ might represent 10 times as many slow electrons as fast electrons, a total range of R of a hundred thousand, yet for $u_1/u_2=0.9$ the maximum gain occurs at $W=9.2$ for $R=0.0001$ and at $W=37$ for $R=10$. The curve of gain *vs.* frequency (Imz *vs.* W) is considerably sharper for the very small values of R. This means that up to a point, greater fractional band widths will be attained in double stream amplifiers as the current in the slower stream is made larger. We see, however, that the curves for $R=10$ (Fig. 6) are sharper than the curves for $R=1$ (Fig. 5) and apparently the latter represents a region of optimum band width.

The envelopes of the curves of Figs. 1–6 are shown in Fig. 7. Here we see that at high frequencies (around $W=10$) Imz varies much less than R. For the highest frequencies shown, Imz changes by a factor of about 30 (34 db) for a change of R by a factor of 100,000.

From (9) we see that gain per meter is proportional to Imz times $\omega\gamma_2$. Consider a case in which W is around 10. First let $\omega_{01}/\omega_{02}=1$. Then let us consider a case in which $\omega_{02}/\omega_{01}=100$ ($R=0.0001$) and let ω_{01} be $\frac{1}{10}$ as large as before. In other words, ω_{02} has been changed by a factor of 10 and ω_{01} has been changed by a factor of $\frac{1}{10}$. From Fig. 3 we see that for the optimum value of u_1/u_2 in each case, Imz decreases by a factor of 0.065, while $\omega_{02}z$ decreases by a factor 10 times this, or 0.65. In this case, the decrease in ω_{02} by a factor of $\frac{1}{10}$ somewhat offsets the increase in ω_{01} by a factor 10, but the change in gain per meter in increasing the charge density of one stream by a factor of 100 and decreasing the charge density in the other stream by a factor of 1/100 is not very large.

One thing we see is that one can obtain "double-stream" gain with very small currents in the slower stream, and presumably with very small currents in the faster stream as well. Thus, one might obtain double stream gain because of primary emission from a negative control grid or secondary or primary emission from a positive control grid or other positive electrode, even in a tube intended to have only a single stream of electrons.

Also, we gather that ion oscillations could occur with a very small ion density.

Figures 8–13 show the ratio phase velocity to the velocity of the faster stream, v/u_2, as a function of frequency W for various ratios of the squares of the plasma frequencies, R, and for various values velocity ratios u_1/u_2. This ratio is given by (10) as (W/Rez).

We see from Fig. 8 that for the very low value of R, $R=0.0001$, the phase velocity is very nearly equal to the

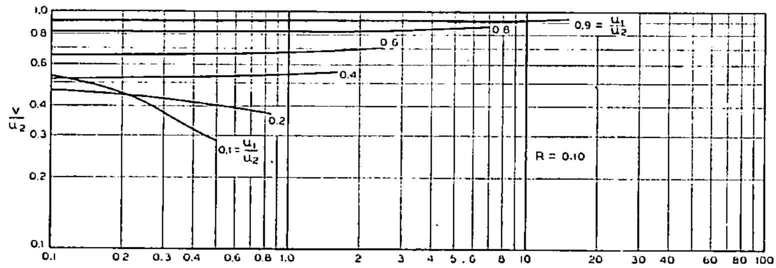


FIG. 11.

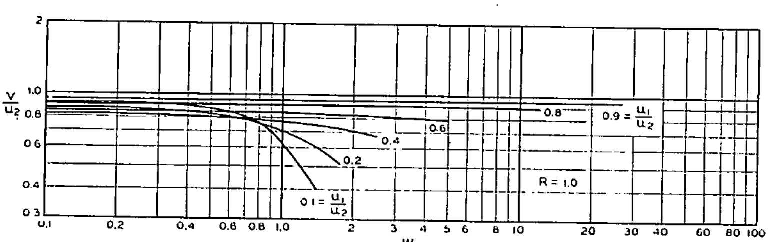


FIG. 12.

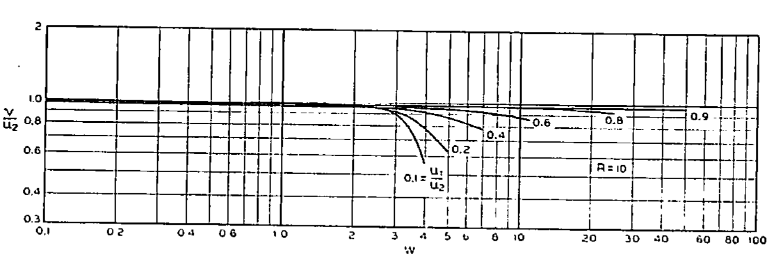


FIG. 13.

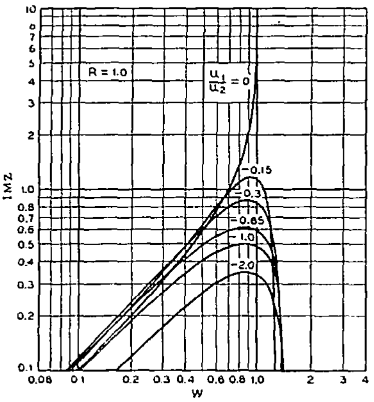


FIG. 14.

velocity of the slower electron stream for all velocity ratios from $u_1/u_2=0.1$ to $u_1/u_2=0.9$. There is a slight upturning as W approaches the limiting frequency at which the gain falls to zero.

While v/u_2 may initially fall as u_1/u_2 is decreased, (12) tells us that v/u_2 must eventually go to unity at $u_1/u_2=0$ and for $W<(R)^{\frac{1}{2}}$ (the region of increasing waves). This is illustrated in the curves for higher values of R. For instance, in Figs. 11 and 12, for the lower values of W v/u_2 finally rises as u_1/u_2 is decreased, after initially falling, and in Fig. 13 ($R=10$) we see that at the lower values of W, v/u_2 is nearly unity for all values of u_1/u_2.

The group velocity v_g is given in terms of the phase velocity v by

$$v_g=v\left(1-\frac{d(\ln v)}{d(\ln\omega)}\right)^{-1}. \tag{14}$$

On a logarithmic plot such as those of Figs. 8–13, the slope is unchanged by the scale factors, and hence $d(\ln v)/d(\ln\omega)$ is just the slope on these figures. On Figs. 8–10 ($R=0.0001$ to $R=0.01$) the slopes are all upward, so that the group velocities are a little larger than the phase velocities. The slope is never as great as unity, however, so the group velocity is always unity and the wave always travels to the right.

In Figs. 11–13 ($R=0.1$ to $R=10$), v/u_2 falls as W is increased, and sometimes rather steeply, giving a group velocity considerably less than the phase velocity. This is particularly true for smaller values of u_1/u_2.

The analysis of the double-stream amplifier for R nearly equal to unity[4] showed the phase velocity to be midway between u_1 and u_2. From Fig. 12 we see that for $R=1$, $u_1/u_2=0.9$, $v/u_2=0.95$, which is in accord with this.

One case is not covered in Figs. 1–13; that in which the two streams of particles travel in opposite directions. Curves for this case for $R=1$ (equal plasma frequencies) are given in Figs. 14 and 15. From Fig. 14 we see that the maximum gain occurs near $W=1$ (the plasma frequency of both streams in this case) and tends to shift to a little lower frequency and a lower value u_1/u_2 is made more negative.

In Fig. 15, let us first consider $-1<u_1/u_2<0$. For these velocity ratios, at low values of W the phase velocity is positive (in the direction of the faster electrons). It rises to infinity at a critical value of W which is a little less than unity, goes to minus infinity as W is increased and remains negative as W is further increased.

The magnitude of the phase velocity increases as u_1/u_2 approaches minus infinity. When u_1 is equal and opposite to u_2, the phase velocity is infinite: the "growing" disturbances merely change exponentially with distance.

At low values of W the group velocity is positive and nearly equal to the phase velocity. As W is increased, the group velocity goes through infinity to negative values where the slope on Fig. 15 is unity, passes through zero where the phase velocity goes to infinity, goes through infinity to negative values where the slope becomes unity again, and approaches the phase velocity, which is negative, for large values of W.

We see that for small values of W the energy flow is to the right. Then in a region a little below $W=1$ the energy flow is to the left. For another region above this the energy flow is to the right again. For still higher frequencies the energy flow is to the left.

We should remember that of two complex values of z which give waves which change with distance, one is the complex conjugate of the other. Thus, whichever way energy flows, there is an increasing wave if there is a complex root.

Figures 14 and 15 show curves for $u_1/u_2=-2$. In this case the faster electrons travel to the left, and the velocities shown in Fig. 15 have signs which in a given region are opposite those for the other curves. This is to be expected. If a wave travels in the direction of the faster electrons, for instance, and the direction of the faster electrons changes from right to left, the direction

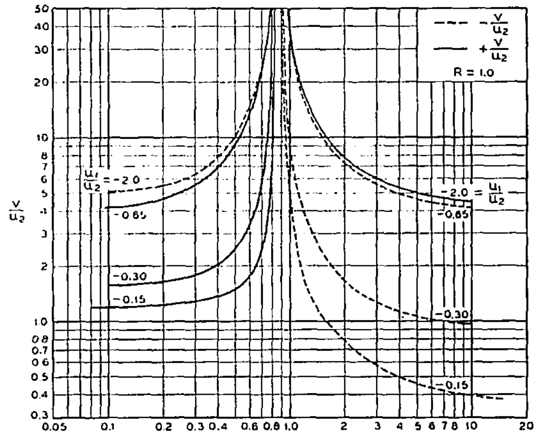


FIG. 15.

of travel of the wave changes. As a matter of fact, the curve for $u_1/u_2=-2$ could be replotted as a curve for $u_1/u_2=-0.5$.

It took a good deal of computing to obtain the curves of Figs. 1–15. It would be even more difficult to explore the more complicated case of a cylindrical beam with a magnetic focusing field, which was discussed in the earlier paper,[1] and no further information about the increasing waves associated with the plasma and cyclotron frequencies in this case have been obtained.

The writer wishes to acknowledge the contributions of Miss M. E. Moore in computing the roots of Eq. (11), and of Dr. L. R. Walker in devising a suitable method of computation.

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